Math 249 Lecture 23 Notes

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1 Combinatorial Species

Last time, we said that a combinatorial species is a functor from the category of finite sets with bijections as morphisms to itself, where the functor takes a set S and associates to it the set of structures of a certain type on S.

1.1 Examples and exponential generating functions

Example 1.1. The functor $L(S) = \{\text{linear orderings of } S\}$ is a species.

Example 1.2. The functor $P(S) = \{\text{permutations of } S\}$ is a species.

Example 1.3. The functor $G(S) = \{\text{graphs with vertex set } S\}$ is a species.

Example 1.4. The functor $\Pi(S) = \{\text{set partitions of } S\}$ is a species.

Definition 1.1. If we have a species A, we associate to it the *exponential generating function* $\mathcal{F}_A$ (sometimes just denoted as A), where

$$\mathcal{F}_A(x) = \sum_{n=0}^{\infty} |A([n])| \frac{x^n}{n!}.$$

Example 1.5. Let L be the species of linear orderings. Then we have

$$L \mapsto L(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}.$$

Example 1.6. Let P be the species of permutations. Then we have

$$P \mapsto P(x) = \frac{1}{1-x}$$

Example 1.7. Let G be the species that takes a set of labeled vertices to the set of graphs on those vertices. Then we have

$$G \mapsto G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}.$$

We can also weight these by the number of edges in the graph.

$$G(x, q) = \sum_{n=0}^{\infty} (1+q)^{\binom{n}{2}} \frac{x^n}{n!}.$$

This is a *mixed generating function* because it is exponential in x and ordinary in q. So the number of graphs with k edges on n labeled vertices is the coefficient of $q^k x^n/n!$.

Example 1.8. Let Π be the species of set partitions. Then we have

$$\Pi \mapsto \Pi(x) = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = e^{e^x - 1},$$

where $B(n)$ is the *Bell number* n.

Definition 1.2. The *trivial* species is $E(S) = \{\varnothing\}$ for all S.

Example 1.9. The exponential generating function for the trivial species E is

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Definition 1.3. An *indicator species* X_k is

$$X_k(S) = \begin{cases} \{\varnothing\} & |S| = k \\ \varnothing & \text{otherwise.} \end{cases}$$

Example 1.10. Here are a few examples of exponential generating functions for indicator species.

$$X_k(x) = \frac{x^k}{k!}$$

$$X_{\neq 0} = e^x - 1$$

$$X_{\text{even}} = \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

1.2 Operations on species

Definition 1.4. Given species A, B, we define *species addition* as

$$(A+B)(S) = A(S) \amalg B(S).$$

We define *species multiplication* as

$$(AB)(S) = \coprod_{S=S_1 \amalg S_2} A(S_1) \times B(S_2).$$

Species addition gives us the collection of structures on S of either structure A or structure B. Species multiplication gives us the collection of structures on S where part of S has structure A and part of S has structure B; we then must take the disjoint union over all ways to split S up into two parts.

Proposition 1.1. *Let A and B be species. Then*

$$\mathcal{F}_{A+B} = \mathcal{F}_A + \mathcal{F}_B,$$
$$\mathcal{F}_{AB} = \mathcal{F}_A \mathcal{F}_B.$$

Proof. Addition follows straightforwardly from the definition:

$$\mathcal{F}_{A+B} = \sum_{n=0}^{\infty} |A([n]) \amalg B([n])| \frac{x^n}{n!} = \sum_{n=0}^{\infty} |A([n])| \frac{x^n}{n!} + \sum_{n=0}^{\infty} |B([n])| \frac{x^n}{n!} = \mathcal{F}_A + \mathcal{F}_B.$$

For multiplication, we have:

$$\mathcal{F}_{AB}(x) = \sum_{n=0}^{\infty} |AB([n])| \frac{x^n}{n!}$$

To find the size of $AB([n])$, we split $[n]$ into the disjoint union of two subsets of sizes k and ℓ, respectively. For each size k, there are $\binom{n}{k}$ ways to partition S, $A([k])$ choices of structures of size k, and $B([\ell])$ choices of structures of size ℓ. The choices of the two structures are independent of each other.

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k+\ell=n} \binom{n}{k} A([k]) B([\ell]) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k+\ell=n} \frac{\cancel{n!}}{k!\ell!} A([k]) B([\ell]) \frac{x^{k+\ell}}{\cancel{n!}}
\end{aligned}$$

Now the inside terms do not depend on n, so we can eliminate the dependence on n in the sums.

$$\begin{aligned}
&= \sum_{k,\ell \geq 0} \frac{A([k])}{k!} \frac{B([\ell])}{\ell!} x^{k+\ell} \\
&= \mathcal{F}_A(x) \mathcal{F}_B(x).
\end{aligned}$$

□

Example 1.11. If we want a linear ordering, we either have the empty set, which has 1 ordering, or we can pick a least element and then apply the ordering to the rest of the set. This gives us

$$L \cong X_0 + X_1 L,$$

$$L(x) = 1 + xL(x).$$

So $L(x) = \frac{1}{1-x}$, as we already know.

Example 1.12. Let $M_k(S) = \{\text{maps } S \to [k]\}$. Then $M_k(S) = E^k$, so

$$M_k(x) = (e^k)^x = e^{kx} = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = \sum_{n=0}^{\infty} k^n \frac{x^n}{n!}.$$

1.3 Stirling numbers

Definition 1.5. The *Stirling number* $S(n, k)$ is the number of partitions of an n-element set into k nonempty blocks.

Example 1.13. Let's compute $S(3, k)$ for different values of k.

$$S(3,0) = 0 \qquad S(3,1) = 1 \qquad S(3,2) = 3 \qquad S(3,3) = 1.$$

Proposition 1.2. *The Stirling numbers satisfy the recurrence relation:*

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

Proof. If we have $n-1$ numbers and we add the number n, we have 2 choices of what to do with n. We can put it in a block by itself, or we can add it to one of the existing blocks. In the first case, we reduce to the number of ways to make $k-1$ blocks with $n-1$ elements. In the second case, there are k choices of which block to place n in, so we have k times the number of ways to make k blocks with $n-1$ elements. □

Let's compute some of the values of $S(n, k)$ using this recurrence relation.

$n \setminus k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

Note that $k!S(n, k) = |\{\text{surjective maps } [n] \to [k]\}|$. So we can define the species $V_k(S) = \{\text{surjective maps } S \to [k]\}$. We get

$$V_k = (X_{\neq 0})^k = (E - X_0)^k,$$
$$V_k(x) = (e^x - 1)^k$$

This shows us something about Stirling numbers:

$$\sum_{n=0}^{\infty} k! S(n,k) \frac{x^n}{n!} = (e^x - 1)^k.$$

If we fix k, we have

$$\sum_{n=0}^{\infty} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{jx}.$$

Taking the coefficient of $x^n/n!$, we get

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$